Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 5, pp. 927–934, 1993

0021–8928/93 $24.00+0.00

0021–8928(93)E0033-S

THE ASYMPTOTIC STABILITY OF MOTION IN SYSTEMS WITH AFTER-EFFECT†

V. S. SERGEYEV

Moscow

(*Received* 21 *February* 1992)

A mechanical system with after-effect is considered, on the assumption that its state at a time t depends not only on its phase coordinates at the time t and on the time itself, but also on the phase coordinates at all previous times from the initial time t_0 on. Lyapunov's first method is used to investigate the stability of motion of such a system. A general solution is constructed for the equations of motion in the neighbourhood of the asymptotically stable trivial solution of the linearized equations, and asymptotic (exponential) stability is proved for the full equations. To demonstrate the application of the method to systems with distributed parameters, the stability of equilibrium of a visco-elastic rod under torsion is considered.

THE CONDITIONS outlined above are satisfied by systems described by Volterra integro-differential equations [1], which may contain non-linear functionals. Fréchet showed [2] that in the analytic case such functionals can be represented by series of multiple integrals. In practical applications one often considers only segments of these series. Equations of a similar kind find application, in particular, in models of visco-elasticity [3–5] and aeroelasticity [6–8].

1. We shall investigate the stability of the motion represented by the trivial solution of the equation

$$\frac{dx}{dt} = A(t)x + \int_{t_0}^{t} K(t,s)x(s)\,ds + F(x,u,t), \quad x \in R^n, \quad u \in R^n \tag{1.1}$$

where the continuous $n \times n$ matrices $A(t)$ and $K(t,s)$ are defined for $t \in I = \{t \in R:\ t \geqslant t_0\}$ and $(t, s) \in J_1' = \{(t,\ s) \in R^2:\ t_0 \leqslant s < t < +\infty\}$, respectively, the vector-valued function $F(x, u, t)$ is analytic in the neighbourhood of the point $x=0$, $u=0$ and does not contain linear and free terms. The coefficients of the expansion $F(x,u,t)$ in powers of $x = \operatorname{col}(x_1, \ldots, x_n)$, $u = \operatorname{col}(u_1, \ldots, u_n)$ are continuous and bounded for $t \in I$; u_i are analytic functionals represented by absolutely convergent Volterra–Fréchet series

$$u_i = \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k=1}^{n} \int_{t_0}^{t} \ldots \int_{t_0}^{t} K_i^{j(k)}(t, s_1, \ldots, s_k) x_{j_1}(s_1) \ldots x_{j_k}(s_k)\, ds_1 \ldots ds_k \tag{1.2}$$

in which $K_i^{j(k)}(t,\ s_1, \ldots,\ s_k)$ are continuous functions defined on the set $J_k' = \{(t,\ s_1, \ldots,\ s_k) \in R^{k+1}:\ t_0 \leqslant s_j < t < +\infty,\ j=1, \ldots, k\}$. In this formula (and throughout this paper) the superscript $j(k)$ denotes the sequence of indices $j_1, \ldots, j_k$. Note that in general the kernels $K(t,s)$, $K_i^{j(k)}(t,\ s_1, \ldots,\ s_n)$ in (1.1), (1.2) will have singularities of the same type as the Abel

†*Prikl. Mat. Mekh.* Vol. 57, No. 5, pp. 166–174, 1993.

kernels on the boundaries of their domains of definition at $s=t$ and $s_j=t$ $(j=1, \ldots, k)$, respectively. To be more definite, we shall assume that the kernels $K_i^{j(k)}(t, s_1, \ldots, s_n)$ in (1.2) satisfy the following limit on the set J_k'

$$|K_i^{j(k)}(t, s_1, \ldots, s_k)| \leq C \frac{\exp[-\alpha_1(t-s_1)-\ldots-\alpha_k(t-s_k)]}{[(t-s_1)\ldots(t-s_k)]^{\rho}} \tag{1.3}$$

where the constants $C>0$, $0 \leq \rho < 1$ are independent of k and $\alpha_p \geq \alpha$ $(p=1, 2, \ldots)$ for some $\alpha > 0$.

Consider the Cauchy problem with initial condition $x_0 = x(t_0)$ $(x_0 = \mathrm{col}(x_{01}, \ldots, x_{0n}))$. Let $X(t, t_0)$ denote a fundamental matrix of solutions of the linearized equation (1.1) such that $X(t_0, t_0) = E_n$.

Theorem 1. Assume that the kernels in Eqs (1.1) and (1.2) satisfy condition (1.3), and moreover

$$\| X(t,s) \| \leq C' \exp[-\beta(t-s)], \quad C', \ \beta > 0 \ -\text{const} \tag{1.4}$$

and let γ be such that $0 < \gamma < \min(\alpha, \beta)$.

Then $\varepsilon > 0$ $(\varepsilon < \gamma)$, $\delta > 0$ exist such that the general solution of Eqs (1.1) and (1.2) in the neighbourhood of zero admits of series expansions

$$x_i(t) = \sum_{m=1}^{\infty} \sum_{m_1+\ldots+m_n=m} S_i^{m(n)}(t) \exp[-(\gamma-\varepsilon)(t-t_0)] x_{01}^{m_1} \ldots x_{01}^{m_n} \tag{1.5}$$

whose coefficients $S_i^{m(n)}(t)$ are continuous and bounded for $t \in I$. The series (1.5) converge absolutely and uniformly for $\|x_0\| < \delta$ for some $\delta > 0$, and the trivial solution is exponentially stable.

Remark. $X(t,s)$ is a matrix of the difference type if A is a constant matrix and $K(t, s) = K'(t-s)$. Condition (1.4), which means that the trivial solution of the linearized equation (1.1) is asymptotically (exponentially) stable, is easily verified, for example, in the case when $K'(t)$ is an exponential polynomial.

The proof of Theorem 1 is based on Lyapunov's first method [9] and proceeds along the lines of the proofs in [10, 11]. We will represent the functionals u_i as series similar to (1.5)

$$u_i = \sum_{m=1}^{\infty} \sum_{m_1+\ldots+m_n=m} P_i^{m(n)}(t) \exp[-(\gamma-\varepsilon)(t-t_0)] x_{01}^{m_1} \ldots x_{0n}^{m_n} \tag{1.6}$$

The coefficients $S_i^{m(n)}(t)$ and $P_i^{m(n)}(t)$ of the series (1.5) and (1.6) are determined successively, using (1.2) and the integral equation [12]

$$x(t) = X(t,t_0) x_0 + \int_{t_0}^{t} X(t,s) F(x(s), u(s), s) ds$$

which is equivalent to Eq. (1.1) with an initial condition. Under these conditions all the functions $S_i^{m(n)}(t)$, $P_i^{m(n)}(t)$ are bounded when $t \in I$.

Suppose that the series v_i and w_i are majorants for x_i and u_i, respectively, and that $F^*(x, y) \gg F(x, y, t)$ for $t \in I$; let C^* be a positive matrix such that

$$X_*(t,s) \leq C^* \exp[-\beta(t-s)]$$

where the elements of $X_*(t, s)$ are taken modulo the corresponding elements of $X(t, s)$. We can then set up majorizing equations [10]

$$w_i = CA\left(\sum_{j=1}^{n} \upsilon_j + B\sum_{j_1,j_2=1}^{n} \upsilon_{j_1}\upsilon_{j_2} + \ldots + B^{k-1}\sum_{j_1,\ldots,j_k=1}^{n} \upsilon_{j_1}\ldots\upsilon_{j_k} + \ldots\right) = \tag{1.7}$$

$$= \frac{CA}{B(1-z)}$$

$$z = B\sum_{j=1}^{n} \upsilon_j, \qquad D(p) = \int_0^\infty \frac{\exp(-ps)}{s^\rho}ds$$

$$\upsilon = C^* x_0 + M^* F^*(\upsilon, w), \qquad A = D(\alpha-\gamma+\varepsilon), \qquad B = D(\alpha) \tag{1.8}$$

$$M^* > \frac{C^*}{\beta-2\gamma+2\varepsilon}(\exp[-(\gamma-\varepsilon)(t-t_0)] - \exp[-(\beta-\gamma+\varepsilon)(t-t_0)])$$

$$\upsilon = \operatorname{col}(\upsilon_1,\ldots,\upsilon_n), \qquad w = \operatorname{col}(w_1,\ldots,w_n)$$

in which M^* is a constant positive matrix.

Equations (1.7) and (1.8) determine υ_i and w_i as an absolutely convergent power series for $\|x_0\| < \delta$. To determine δ one can proceed as in [13].

An analogue of Theorem 1 for systems with distributed parameters is the assertion stated in Sec. 2, concerning the stability of equilibrium of a visco-elastic rod under torsion.

2. Consider a thin visco-elastic rod of length l, one end of which is clamped and the other free, under torsion around an axis Oz where O is the left end of the rod.

We shall assume that the axis of the rod is not deformed, that the cross-sections of the rod remain plane and that displacements of the points in a cross-section may occur only due to rotation of the cross-section as a whole. The distribution of masses over all cross-sections orthogonal to the rod axis is assumed to be the same.

The rod is subject to external body forces whose moment in each cross-section depends only on the angle of torsion θ of that section. The lateral surface is free of the action of the external forces. The moment of the visco-elastic forces depends on $\partial\theta/\partial z$ and is determined by a linear Volterra integral operator [1]

$$M(\theta, z, t) = k_1\frac{\partial\theta(z,t)}{\partial z} + \int_{t_0}^{t} k_2(t-s)\frac{\partial\theta(z,s)}{\partial z}ds$$

where k_1 is the (constant) torsional rigidity, and the relaxation kernel $k_2(t)$ is continuous in $t \in I$. The equation of motion, derivable by using the Hamilton–Ostrogradskii principle for systems with distributed parameters [14], is

$$I_1\frac{\partial^2\theta(z,t)}{\partial t^2} - k_1\frac{\partial^2\theta(z,t)}{\partial z^2} - \int_{t_0}^{t} k_2(t-s)\frac{\partial^2\theta(z,s)}{\partial z^2}ds - f(\theta) = 0 \tag{2.1}$$

where I_1 is the moment of inertia of a rod of unit length about its axis, and $f(\theta)$ is the moment of the external forces. We shall assume that $f(\theta)$ is an odd function and can be expanded in a convergent power series in the neighbourhood of zero, so that

$$f(\theta) = \sum_{n=1}^{\infty} a_{2n-1}\theta^{2n-1}, \qquad a_{2n-1} = \text{const}$$

If, for example, the rod is in a horizontal position in the field of gravity and the distance between the centre of mass of the cross-section and the rod axis is r, then $f(\theta) = -mgr\sin\theta$, where mg is the weight of a rod of unit length.

We shall investigate the stability of the equilibrium position corresponding to the trivial solution of Eq. (2.1), relative to perturbations at time $t = t_0$ of the angle (θ, x, t) and its velocity, subject to the boundary conditions

$$\theta(0,t) = \partial\theta(z,t)/\partial z|_{z=l} = 0 \tag{2.2}$$

To that end, we put $\theta = \mu U$ ($\mu = \text{const} \ll 1$) and construct the general solution of Eq. (2.1) in the neighbourhood of zero as a series

$$U(z,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu^{2n-2}\sin(\alpha_k z)T_k^{(2n-1)}(t) \tag{2.3}$$

where, by virtue of our boundary conditions, we have put $\alpha_k = \pi(2k-1)/(2l)$. Substituting the series (2.3) into (2.1), we see that, by the properties of Eq. (2.1), a solution may be obtained—at least, formally—in the required form. When that is done the functions $T_k^{(2n-1)}(t)$ (k, $n = 1, 2, \ldots$) will be solutions of the equations

$$I_1\frac{d^2T_k^{(2n-1)}(t)}{dt^2} + (k_1\alpha_k^2 - a_1)T_k^{(2n-1)}(t) + \\ + \alpha_k^2\int_{t_0}^{t} k_2(t-s)T_k^{(2n-1)}(s)ds = S_k^{(2n-3)}(t) \tag{2.4}$$

where $S_k^{(-1)}(t) \equiv 0$ and $S_k^{(2n-3)}(t)$ ($n = 2, 3, \ldots$) are known continuous functions, if all the functions $S_k^{(2m-3)}(t)$ are known for $m < n$ and $k = 1, 2, \ldots$.

Let us assume that the initial values $\theta(z, t_0) = \mu\varphi(z)$ and $\partial\theta(z, t)/\partial t|_{t=t_0} = \mu\psi(z)$ are sufficiently smooth functions, expanded in Fourier series over the interval $[0, l]$ with respect to the system of functions $\sin(\alpha_k z)$. Let

$$b_k = \frac{2}{l}\int_0^l \varphi(x)\sin(\alpha_k x)dx, \qquad c_k = \frac{2}{l}\int_0^l \psi(x)\sin(\alpha_k x)dx$$

be the corresponding Fourier coefficients. Let us consider solutions of Eqs (2.4) for $n = 1$ with initial conditions $T_k^{(1)}(t_0) = b_k$, $dT_k^{(1)}(t)/dt|_{t=t_0} = c_k$. The solution of these equations for $k = 1, 2, \ldots$ may be written in the form

$$T_k^{(1)}(t) = x_{11}^{(k)}(t)b_k + x_{12}^{(k)}(t)c_k, \quad dT_k^{(1)}(t)/dt = x_{21}^{(k)}(t)b_k + x_{22}^{(k)}(t)c_k$$

where the fundamental matrix $(x_{ij}^{(k)}(t))$ is such that $(x_{ij}^{(k)}(t_0)) = E_2$.

Taking the dissipative properties of visco-elastic materials into account, we will make the following assumption as to the behaviour of the functions $x_{ij}^{(k)}(t)$: for all $k = 1, 2, \ldots$, constants $M > 0$, $\lambda_0 > 0$ exist independent of k, such that for $t \in I$

$$|x_{11}^{(k)}(t)|,\ |x_{12}^{(k)}(t)|,\ |x_{22}^{(k)}(t)| \leqslant M\exp[-\lambda_0(t-t_0)] \tag{2.5}$$

We shall also assume that the following conditions hold for the integral kernel $K_2(t)$

$$|K_2(t)| \leqslant \kappa_1\exp(-\beta_1 t), \quad \beta_1 - \kappa_1/k_1 > 0; \quad \kappa_1 > 0,\ \beta_1 > 0 - \text{const} \tag{2.6}$$

We will give a simple example to show how the validity of conditions (2.5) may be related to the parameters of Eq. (2.4) for $n = 1$. Let Eq. (2.4) be

$$\frac{d^2T_k^{(1)}}{dt^2} + \alpha_k^2(T_k^{(1)} + \int_{t_0}^{t} k_2(t-s)T_k^{(1)}(s)ds) = 0$$

$$k_2(t) = \kappa\exp(-\beta t)$$

$$\kappa,\ \beta\ -\ \text{const}$$

It follows from an analysis of the characteristic equation [12] that if $\beta>0$, $\kappa<0$, $\beta+\kappa>0$, its roots will have negative real parts for all real α_k and the solution of the equation will be exponentially stable. One can then find constants M, $\lambda_0>0$ such that conditions (2.5) hold for all $k=1, 2, \ldots$. In that situation $|x_{12}^{(k)}(t)|\leqslant C_1/\alpha_k$ and $|x_{21}^{(k)}(t)|\leqslant \alpha_k/C_2$, where $C_1>0$, $C_2>0$ are constants independent of k.

Theorem 2. If conditions (2.5) and (2.6) hold, the equilibrium position of the visco-elastic rod corresponding to the trivial solution of Eq. (2.1) with boundary conditions (2.2) is asymptotically stable in Lyapunov's sense with respect to perturbations of the initial conditions such that the functions $\varphi(z)$, $\psi(z)\in C^3$ and $\varphi'''(z)$, $\psi'''(z)$ are of bounded variation. The angle of torsion in the perturbed motion is such that $\theta(z, t)\to 0$ exponentially as $t\to+\infty$ for each $z\in[0, 1]$, and the general solution of the problem is defined by a series (2.3) in which the function $T_k^{2n-1}(t)$, which tends exponentially to zero, depends on the constant Fourier coefficients b_m and c_m of the functions $\varphi(z)$ and $\psi(z)$ in $[0, l]$ relative to the system of functions $\sin(\alpha_k z)$. The series (2.3) is a solution of Eq. (2.1) and converges absolutely and uniformly for all $z\in[0, l]$, $t\in I$ and μ, b_m and c_m such that $\mu\sum_{k=1}^{\infty}(|b_k|+|c_k|)<\delta$ for some $\delta>0$.

Remark. Let us extend each of the functions $\varphi(z)$ and $\psi(z)$ to the interval $[l, 2l]$ in such a way that the extended functions are even with respect to the straight line $z=l$, and then extend it further to the interval $[-2l, 0]$ so that the extension is an odd function (relative to zero). One can then define these functions over the entire real axis as $4l$-periodic functions. The restrictions imposed on $\varphi(z)$, $\psi(z)$ and their derivatives in Theorem 2, the boundary conditions and Eq. (2.1) itself imply that these periodic functions will have everywhere continuous third derivatives of bounded variation. Hence their Fourier coefficients satisfy the limit [15]

$$b_k = O\left(\frac{1}{k^4}\right), \qquad c_k = O\left(\frac{1}{k^4}\right) \tag{2.7}$$

By (2.7), the solution of the linearized equation (2.1), expressed as a series, and the series for the first and second derivatives of $U(x, t)$, converge absolutely and uniformly, so the linearized equation in the required form indeed has a solution.

Proof of Theorem 2. Suppose that all the functions $T_k^{(2m-1)}(t)$, where $1<m<n$ and $k=1, 2, \ldots$, are determined from Eq. (2.4) with initial conditions $T_k^{(2m-1)}(t_0)=0$, $dT_k^{(2m-1)}(t)/dt\,|_{t=t_0}=0$ and satisfy the inequalities

$$|T_k^{(2m-1)}(t)| \leqslant C\exp[-\lambda_0(t-t_0)], \qquad C=\text{const}>0 \tag{2.8}$$
$$m = 1, 2, \ldots, n-1$$

Suppose that $T_k^{(2m-1)}(t)$, as series in the parameters b_q, c_r $(q, r=1, 2, \ldots)$ are absolutely convergent. To determine $T_k^{(2n-1)}(t)$ we have Eq. (2.4), in which $S_k^{(2n-3)}(t)$ is, for all fixed k, n, a series in $a_{2i}+1$ $(i=1, 2, \ldots, n-1)$ and $T_p^{(2s-1)}(t)$ $(s=1, 2, \ldots, n-1;\ p=1, 2, \ldots)$ with rational coefficients. The series $S_k^{(2n-3)}(t)$ in the parameters b_q and c_r is absolutely convergent, and by (2.8) and the properties of $f(\theta)$, we have the limit

$$|S_k^{(2n-3)}(t)| \leqslant K^{(2n-3)}\exp[-3\lambda_0(t-t_0)] \tag{2.9}$$

where $K^{(2n-3)}>0$ is a constant that depends on $K^{(2m-3)}$ for $m<n$. The solution of Eq. (2.4) with initial conditions $T_k^{(2n-1)}(t_0)=0$, $dT_k^{(2n-1)}(t)/dt\,|_{t=t_0}=0$ is given by the formula

$$T_k^{(2n-1)}(t) = \int_{t_0}^{t} x_{12}^{(k)}(t-s)S_k^{(2n-3)}(s)\,ds \tag{2.10}$$

from which, using (2.5) and (2.9), we obtain the limit

$$|T_k^{(2n-1)}(t)| \leqslant MK^{(2n-3)}(2\lambda_0)^{-1}\exp[-\lambda_0(t-t_0)]$$

which in turn implies that the series for $T_k^{(2n-1)}(t)$ is absolutely convergent.

Let V denote a majorant for the series (2.3), considered as an expansion in powers of μ, b_p and c_s. Applying the procedure used in Sec. 1, we set up an equation for V

$$V = M\rho + \frac{M}{\lambda_0} f^*(V), \qquad \rho = \sum_{k=1}^{\infty}(|b_k|+|c_k|) \tag{2.11}$$

where $f^*(\theta)$ is a majorant of the series for $f(\theta)-a_1\theta$.

Equation (2.11) defines V as a series in the parameter ρ, whose radius of convergence ρ_0 is finite. For μ, b_k and c_k such that $\rho \leqslant \rho_0$, the series (2.3) is absolutely and uniformly convergent for $z \in [0, l]$, $t \in I$. One can also show by constructing majorants that the series for $\partial^2\theta(z, t)/\partial z^2$ and $\partial^2\theta(z, t)/\partial t^2$ are absolutely and uniformly convergent.

The relation

$$\frac{dT_k^{(2n-1)}(t)}{dt} = \int_{t_0}^{t} x_{22}^{(k)}(t-s)\, S_k^{(2n-1)}(s)\, ds, \quad n>1$$

which is analogous to (2.10), implies

$$\frac{\partial^2 T_k^{(2n-1)}(t)}{\partial t^2} = x_{22}^{(k)}(t-t_0)\, S_k^{(2n-1)}(t_0) + \int_{t_0}^{t} x_{22}^{(k)}(t-s)\frac{d}{ds} S_k^{(2n-3)}(s)\, ds$$

on whose basis one constructs a majorant series for $\partial^2\theta(z, t)/\partial t^2$, similar to the series on the right of (2.11), which is absolutely convergent provided that (2.7) and (2.5) hold.

Equation (2.1), with the solution determined for $\theta(z, t)$ as Fourier series substituted into it, may be treated as a Volterra integral equation for the unknown $\partial^2\theta(z, t)/\partial t^2$. By virtue of the above, the right-hand side $\Phi(z, t)$ of this integral equation, expressed in terms of $f(\theta)$ and $\partial^2\theta(z, t)/\partial t^2$, will be finite. The solution of the integral equation, constructed using the resolvent $\Gamma(t, s)$ [3, 16] for the kernel $k_2(t-s)$, is

$$\frac{\partial^2\theta(z,t)}{\partial z^2} = \Phi(z,t) + \int_{t_0}^{t}\Gamma(t,s)\,\Phi(z,s)\,ds \tag{2.12}$$

$$\Phi(z,t) = \frac{l_1}{k_1}\frac{\partial^2\theta(z,t)}{\partial t^2} - \frac{f(\theta)}{k_1}$$

If (2.6) is true, the right-hand side of (2.12) is a bounded function and $\partial^2\theta(z, t)/\partial z^2$ is an absolutely and uniformly convergent series for $z \in [0, l]$, $t \in I$.

Remark. The estimate (2.7) will also hold under slightly weaker assumptions concerning the properties of the functions $\varphi(z)$ and $\psi(z)$ than those of Theorem 2. It is sufficient to require that $\varphi'''(z)$, $\psi'''(z)$ be of bounded variation or absolutely continuous in $[0, l]$.

If $k_2(t)$ is an exponential polynomial [12], it can be shown that, for the majorizing series for $\partial^2\theta(z, t)/\partial t^2$ and $\partial^2\theta(z, t)/\partial t^2$ to converge, the function $\psi(z)$ must be such that $c_k = O(1/k^3)$. This means that $\varphi'''(z)$ and $\psi'''(z)$ in the formulation of Theorem 2 must be of bounded variation or absolutely continuous in $[0, l]$.

Series of type (2.3), which depend on arbitrary constants b_p, c_q $(p,\ q = 1, 2, \ldots)$, may be considered for systems with distributed parameters as analogues of the series of Lyapunov's first method.

Note that in order to investigate the stability of systems with after-effect or systems of processes with either lumped or distributed parameters, one often uses the method of Lyapunov functionals or functions [17, 21], characterizing the process by one or two metrics. A comprehensive survey of research on stability in such systems may be found in the above-mentioned monographs, and also in [12, 22–24].

When investigating the torsion of a visco-elastic rod, a good measure of the initial perturbation (the deviation of the functions $\mu\varphi(z), \mu\psi(z)$ and their first and second derivatives) is the following series, which is convergent under the assumptions of Theorem 2

$$\rho_0 = \sum_{n=1}^{\infty} \mu(1+n+n^2)(|a_n|+|b_n|)$$

Then, as follows from Theorem 2, for each fixed $z \in [0, l]$ one has exponential stability relative to the functions $\theta(z, t)$, $\partial^k\theta(z, t)/\partial z^s \partial t^p$, $k = s+p = 1, 2; s, p \geqslant 0, s \neq p$.

In connection with Theorem 2, it should be mentioned that a related problem—the determination of perturbed motion for a given steady-state of a thermovisco-elastic system, represented by formal power series in parameters characterizing the applied perturbing forces, and the derivation of the spectral conditions of asymptotic stability relative to the perturbations—was treated in [25, 26].

REFERENCES

1. VOLTERRA V., *Theory of Functionals and Integral and Integrodifferential Equations.* Nauka, Moscow, 1982.
2. FRECHET M., Sur les fonctionelles continues. *Ann. École Normale Sup., Sér.* **3**, 27, 193–216, 1910.
3. VOLTERRA V., Sur les équations integro-différentielles et leurs applications. *Acta Math.* **35**, 4, 295–354, 1912.
4. RABOTNOV Yu. N., *Elements of the Hereditary Mechanics of Solids.* Nauka, Moscow, 1977.
5. IL'YUSHIN A. A. and POBEDRYA B. E., *Foundations of the Mathematical Theory of Thermovisco-elasticity.* Nauka, Moscow, 1970.
6. BELOTSERKOVSKII S. M., SKRIPACH B. K. and TABACHNIKOV V. G., *The Wing in Unsteady Gas Flow.* Nauka, Moscow, 1971.
7. MOROZOV V. I., Mathematical models of the dynamics of an aeroelastic aircraft. In *Computer-aided Research in Aircraft Engineering,* No. 1310, pp. 39–51. Trudy Vysshei Voyenno-Inzh. Akad. im. N. E. Zhukovskogo, 1981.
8. ASTAPOV I. S., BELOTSERKOVSKII S. M. and MOROZOV V. I., Non-linear integrodifferential equations of aeroelasticity. *Izv. Akad. Nauk SSSR. MTT* 6, 61–70, 1985.
9. LYAPUNOV A. M., General problem of the stability of motion. In *Collected Papers,* Vol. 2, pp. 7–263. Izd. Akad. Nauk SSSR, Moscow, 1956.
10. SERGEYEV V. S., On the stability of solutions of integrodifferential equations in certain cases. In *The Method of Lyapunov Functions in the Analysis of System Dynamics*, pp. 98–105. Nauka, Novosibirsk, 1987.
11. SERGEYEV V. S., On the stability of solutions for a certain class of integrodifferential equations. *Diff. Urav.* **22,** 3, 519–523, 1986.
12. BYKOV Ya. V., *On Some Problems of the Theory of Integrodifferential Equations.* Izd. Akad. Nauk KirgSSR, 1957.
13. SERGEYEV V. S., On estimating the domain of attraction for a certain class of integrodifferential equations. In *Stability of Motion*, pp. 88–93. Nauka, Novosibirsk, 1985.
14. MOISEYEV N. N. and RUMYANTSEV V. V., *Dynamics of a Body with Cavities Containing a Liquid.* Nauka, Moscow, 1965.
15. BARI N. K., *Trigonometric Series*. Fizmatgiz, Moscow, 1961.
16. MIKHLIN N. N., *Integral Equations*. Gostekhizdat, Moscow, 1947.
17. KRASOVSKII N. N., *Some Problems of the Theory of Stability of Motion.* Fizmatgiz, Moscow, 1959.
18. KOLMANOVSKII V. B. and NOSOV V. R., *Stability and Periodic Regimes of Controllable Systems with After-effect*. Nauka, Moscow 1981.
19. MATROSOV V. M., ANAPOL'SKII L. Yu. and VASIL'YEV S. N., *The Comparison Method in the Mathematical Theory of Systems.* Nauka, Novosibirsk, 1980.
20. SIRAZETDINOV T. K., *Stability of Systems with Distributed Parameters.* Nauka, Novosibirsk, 1987.
21. MOVCHAN A. A., Stability of processes relative to two metrics. *Prikl. Mat. Mekh.* **24,** 6, 988–1001, 1960.
22. BYKOV Ya. V., Some problems of the qualitative theory of integrodifferential equations. *Issledovaniya po Integrodiff. Uravn. v Kirgizii, Izd. Akad. Nauk KirgSSSR, Frunze* 1, 3–54, 1961.
23. MYSHKIS A. D., Some problems of the theory of differential equations with deviating argument. *Usp. Matem. Nauk* **32,** 2, 173–202, 1977.
24. AZBELEV N. V., MAKSIMOV V. P. and RAKHMATULLINA L. F., *Introduction to the Theory of Functional-differential Equations*. Nauka, Moscow, 1991.

25. GROMOV V. G., A dynamical stability criterion and the supercritical behaviour of flexible visco-elastic bodies under a thermomechanical load. *Dokl. Akad. Nauk SSSR* **220**, 4, 805–808, 1975.
26. GROMOV V. G., Lyapunov's first method in the dynamical stability of flexible thermovisco-elastic bodies. *Dokl. Akad. Nauk SSSR* **223**, 4, 819–822, 1975.

Translated by D.L.